Tutorial 3:

- Angular momentum;
- Hydrogen atom
- Spin half particles, properties of Pauli spin matrices.

Section A - Angular Momentum and Hydrogen Atom

Q1:

10

Using the definition $\vec{L} = \vec{r} \times \vec{p}$ of the orbital angular momentum operator, evaluate $[L_x, L_y]$.

10

2013

Q2:

Using the commutation relations

$$[x, p_x] = [y, p_y] = [z, p_z] = i\hbar,$$

deduce the commutation relation between the components of angular momentum operator L.

$$[L_x, L_y] = i\hbar L_z$$

$$[L_y, L_z] = i\hbar L_x \text{ and}$$

$$[L_z, L_x] = i\hbar L_y.$$

20

2014

Q3:

Using dimensional analysis, explain why the angular momentum of a particle cannot be $\hbar^2$.

10

2015

Q4:

If $\hat{x}$ and $\hat{p}$ are the position and momentum operators, prove the commutation relation $[\hat{p}^2, \hat{x}] = -2i\hbar p$.

10

2015

Q5:

Find the energy, momentum and wavelength of a photon emitted by a hydrogen atom making a direct transition from an excited state with n = 10 to the ground state. Also find the recoil speed of the hydrogen atom in this process. *10*

2016

Q6:

Evaluate the most probable distance of the electron from nucleus of a hydrogen atom in its $2p$ state. What is the probability of finding the electron at this distance ? 20

2017

Q7:

Explain why the square of the angular momentum (L^2) and only one of the components (L_x, L_y, L_z) of L are regarded as constants of motion. 15

2017

Q8:

The ground state wave function for hydrogen atom is

$$\psi(r) = \frac{1}{\sqrt{\pi a_0^3}} e^{-r/a_0}$$

where a_0 is the Bohr radius. Sketch the wave function and the probability density as a function of the separation distance r. Calculate the probability that the electron in the ground state is found beyond the Bohr radius. 20

2018

Q9:

Define mathematically the Bohr radius of a hydrogen atom and show that the binding energy at state n of this atom can be given by

$$E_n = -\frac{1}{2}\frac{Ze^2}{(a/Z)}\frac{1}{4n^2\pi\varepsilon_0}$$

where Z is the atomic number of H atom. Calculate the numerical values of a and E_1 of H atom. 15

2019

Q10:

Define angular momentum of a particle and find out the three components of the angular momentum operator $\hat{L}$ in Cartesian coordinates. Show that

$$\hat{L}^2 = -\hbar^2\left[r^2\nabla^2 - \frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)\right]$$

Prove that the operator $\hat{L}^2$ can also be expressed as

$$\hat{L}^2 = -\hbar^2\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]$$

in spherical polar coordinates (r, θ, ϕ). 20

2019

Q11.

Prove that Bohr hydrogen atom approaches classical conditions, when n becomes very large and small quantum jumps are involved. 10

2020

Q12:

Consider a Hermitian operator A with property $A^3 = 1$. Show that $A = 1$. 15

2020

Q13:

Prove the commutation relation for the angular momentum :

$$[L^2, L_z] = 0$$

Also show that $(\vec{L} \times \vec{L}) = i\hbar\vec{L}$. 15

2020

Q14:

Calculate the frequency of the electron in the first Bohr orbit of hydrogen atom. 15

2020

Q15:

The raising (J_+) and lowering (J_-) operators are defined by $J_+ = J_x + iJ_y$ and $J_- = J_x - iJ_y$ respectively. Prove the following identities :

(i) $[J_z, J_\pm] = \pm\hbar J_\pm$

(ii) $J_- J_+ = J^2 - J_z^2 - \hbar J_z$ 10

2022

Q16:

Show that $E_n = \langle V \rangle$ in the stationary states of the hydrogen atom. 15

2022

Q17:

Evaluate the most probable distance of the electron of the hydrogen atom in its $2p$ state. What is the radial probability density at that distance ? 15

2023

Q18.

Show that the square of the orbital angular momentum operator (L^2) commutes with any of the components of angular momentum operator L.

Is it possible to measure L^2, L_x, L_y and L_z simultaneously ? Give reasons for your answer. *6+4=10*

2024

Q19.

Prove that :

(i) $[L^2, L_z] = 0$

(ii) $[L_z, L_+] = \hbar L_+$

(iii) $[L_+, L_-] = 2\hbar L_z$

(iv) $L_+ L_y = L^2 - L_z^2 + \hbar L_z$

where $\hbar = \frac{h}{2\pi}$ (h is Planck's constant) *5+5+5+5=20*

2024

Q20.

By applying the Schrödinger's equation to the ground state of hydrogen atom, determine the zero-point energy. 2025 15

Section B- Spin Half Particles, Pauli Spin Matrices

Q1:

Write down Pauli spin matrices. Express J_x, J_y and J_z in terms of Pauli spin matrices. 10

2014

Q2:

Write down the matrix representation of the three Pauli matrices σ_x, σ_y and σ_z. Prove that these matrices satisfy the following identities :

(i) $[\sigma_x, \sigma_y] = 2\,i\,\sigma_z$

(ii) $[\sigma^2 . \sigma_x] = 0$

(iii) $(\vec{\sigma} . \vec{A})(\vec{\sigma} . \vec{B}) = \vec{A} . \vec{B} + i\,\vec{\sigma} . (\vec{A} \times \vec{B})$

if $\vec{A}$ and $\vec{B}$ commute with Pauli matrices. *8+4+4+4=20*

2016

Q3:

Prove the following identities :

(i) $[\hat{p}_x, \hat{L}_y] = i\hbar\,\hat{p}_z$ 7

(ii) $e^{i(\hat{\sigma}\cdot\hat{n})\theta} = \cos\theta + i(\hat{\sigma}\cdot\hat{n})\sin\theta$ 8

2018

Q4:

Define Pauli spin matrices σ_x, σ_y and σ_z. Using these definitions, prove the following :

(i) $\sigma_x^2 = \sigma_y^2 = \sigma_z^2 = 1$

(ii) $\sigma_x\sigma_y = i\sigma_z$; $\sigma_z\sigma_x = i\sigma_y$; $\sigma_y\sigma_z = i\sigma_x$ 15

2019

Q5:

If the z-component of an electron spin is $+\frac{\hbar}{2}$, what is the probability that its component along a direction z' (forming an angle θ with z-axis) is $\frac{\hbar}{2}$ or $-\frac{\hbar}{2}$? What is the average value of spin along z'? 20

2020

Q6:

Using Pauli spin matrices prove that,

(i) $\sigma_x \sigma_y + \sigma_y \sigma_x = 0$; $\sigma_y \sigma_z + \sigma_z \sigma_y = 0$; $\sigma_x \sigma_z + \sigma_z \sigma_x = 0$

(ii) $\sigma_+ \sigma_- = 2(1+\sigma_z)$

(iii) $\sigma_\alpha + \sigma_\beta = i\sigma_\gamma$ where $\alpha \neq \beta \neq \gamma$ 8+6+6

2021

Q7:

What is the spin wave function (for $s = \frac{1}{2}$) if the spin component in the direction of unit vector η has a value of $\frac{1}{2}\hbar$? 15

2022

Q8:

Obtain the normalized eigenvectors of σ_x and σ_y matrices. 15

2022

Q9:

An operator P describing the interaction of two spin $\frac{1}{2}$ particles is $P = a + b\vec{\sigma}_1 \cdot \vec{\sigma}_2$, where a and b are constants, and $\vec{\sigma}_1$ and $\vec{\sigma}_2$ are Pauli matrices of the two spins. The total spin angular momentum $\vec{S} = \vec{S}_1 + \vec{S}_2 = \frac{1}{2}\hbar(\vec{\sigma}_1 + \vec{\sigma}_2)$. Show that P, S^2 and S_z can be measured simultaneously. 15

2023

Q10.

State how for spin-half particles, the spin (σ) can be expressed by its three components σ_x, σ_y and σ_z. 20

2025